where

$$[\beta] = \begin{bmatrix} 1 - \rho_1 F_{11} & -\rho_1 F_{12} \ldots & -\rho_1 F_{1n} \\ -\rho_2 F_{21} & 1 - \rho_2 F_{22} \ldots & -\rho_2 F_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -\rho_n F_{n1} & -\rho_n F_{n2} \ldots & 1 - \rho_n F_{nn} \end{bmatrix}$$

Solution of Eq. (6) reveals that

$$J_k = a_{kj}\, \epsilon_j \tag{7}$$

where a_{kj} (the kth row, jth column) is the element of the inverse matrix $[\beta]^{-1}$. Substitution of (7) into (4) yields

$$\mathfrak{F}_{jk} = a_{kj}\, \epsilon_k\, A_k\, \epsilon_j / \rho_k\, A_j \tag{8}$$

By the direct expansion of $[\beta]^{-1}$, it is established readily that

$$a_{kj}\, A_k / \rho_k = a_{jk}\, A_j / \rho_j$$

thus

$$\mathfrak{F}_{jk} = a_{jk}\, (\epsilon_j\, \epsilon_k / \rho_j)$$

This technique readily yields script F by a single matrix inversion. The method lends itself well to problems handled analytically with n small or for problems solved by the use of a digital computer with n large.

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Motion of an Asymmetric Spinning Body with Internal Dissipation

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Introduction

The moment-free motion of a spinning body with internal energy dissipation is important in satellite wobble-damping problems. The important relationship for such problems is that between the kinetic energy of rotation T of the body and the half-cone angle θ of precession. For a symmetric body this relationship is well known.[1, 2]

The mathematical solution for the moment-free motion of an asymmetric spinning body results in elliptic functions and may be found, for example, in Ref. 3. The purpose of this note is to put the classical results for the attitude drift of an asymmetric body into the form most convenient for treating the problems of internal energy dissipation and to show that, when formulated properly, the results take on a form very similar to those for the symmetric case.

Theory

For a nearly rigid symmetric body with principal axes 1, 2, 3 and moments of inertia I, I, J (Fig. 1) the attitude angle θ between the spin axis 3 and the constant angular momentum vector h can be expressed in terms of the kinetic energy T by starting with the momentum and energy equations

$$h^2 = I^2(\omega_1{}^2 + \omega_2{}^2) + J^2\omega_3{}^2 \tag{1}$$

$$2T = I(\omega_1{}^2 + \omega_2{}^2) + J\omega_3{}^2 \tag{2}$$

and noting that $\cos\theta = J\omega_3/h$.

The energy and momentum associated with the internal motion is neglected. The desired relationship is then

$$\cos^2\theta = [(h^2 - 2TI)J/(J - I)h^2] \tag{3}$$

which upon differentiation results in the equation

$$(2 \sin\theta \cos\theta)\dot{\theta} = -[2\dot{T}IJ/(I - J)h^2] \tag{4}$$

Since $\dot{T}$ for energy dissipation is negative, $\dot{\theta}$ must be negative for $J > I$ and positive for $J < I$.

To solve for θ as a function of time the small energy dissipation rate $\dot{T}$ is obtained from an examination of the internal dissipative mechanisms within the body. The technique used is outlined for particular cases in References 1, 2, 4 and 5. For the purposes of this discussion $\dot{T}$ may be assumed known.

For the asymmetric body with moments of inertia $A, B, C,$ about principal axes 1, 2, 3, the solutions for the angular velocities are in terms of elliptic functions. Assume $A > B > C$ and $h^2 < 2TB$. This corresponds to the initial conditions for a long, thin body spinning about its long axis Then the angular velocities are given as:[3]

$$\omega_1 = \left[\frac{h^2 - 2TC}{A(A - C)}\right]^{1/2} cn\; p(t - t_0)$$

$$\omega_2 = \left[\frac{h^2 - 2TC}{B(B - C)}\right]^{1/2} sn\; p(t - t_0)$$

$$\omega_3 = -\left[\frac{2TA - h^2}{C(A - C)}\right]^{1/2} dn\; p(t - t_0)$$

$$p = \left[\frac{(B - C)(2TA - h^2)}{ABC}\right]^{1/2} \tag{5}$$

and the modulus k of the elliptic functions is

$$k = \left[\frac{(A - B)}{(B - C)}\,\frac{h^2 - 2CT}{2AT - h^2}\right]^{1/2}$$

The properties of the elliptic functions indicate that ω_1 and ω_2 are oscillatory with zero mean value, while ω_3 fluctuates slightly but is never zero. The motion is then a spin about axis 3 with a slight wobble.

On eliminating ω_1 from the momentum and energy equations, one can write

$$\left(\frac{C\omega_3}{h}\right)^2 = \cos^2\theta = \frac{(2TA - h^2)C}{(A - C)h^2} \times \left[1 - \frac{(A - B)(h^2 - 2TC)}{(B - C)(2TA - h^2)}\, sn^2\, p(t - t_0)\right] \tag{6}$$

Since $sn^2\, p(t - t_0)$ oscillates between one and zero, Eq. (6) indicates that the nutation or cone angle θ, which is not oscillatory for a symmetric body, fluctuates with frequency $2p$ for an asymmetric body. If one chooses as new variables the maxi-

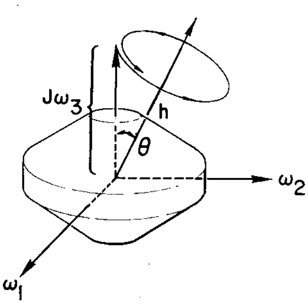

Fig. 1 Motion of free spinning body

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mum and minimum values reached by θ on each cycle, θ_{max} and θ_{min}, one obtains

$$\cos^2\theta_{max} = \frac{C}{(B - C)h^2}(2TB - h^2) \tag{7}$$

$$\cos^2\theta_{min} = \frac{C}{(A - C)h^2}(2TA - h^2) \tag{8}$$

After differentiating, the following equations are obtained:

$$2\sin\theta_{max}\cos\theta_{max}\dot{\theta}_{max} = -\frac{2\dot{T}CB}{(B - C)h^2} \tag{9}$$

$$2\sin\theta_{min}\cos\theta_{min}\dot{\theta}_{min} = -\frac{2\dot{T}CA}{(A - C)h^2} \tag{10}$$

Comparison of Eq. (9) and (10) with (4) indicates that the form is identical with Eq. (4), if B and A, respectively, are substituted for I, and C for J. The similarity of form when θ_{max} or θ_{min} is used as a variable indicates that the techniques used for the symmetric case (e.g., in Refs. 1, 2, 4, and 5) can be applied with little difficulty to the asymmetric case. Of course, the asymmetry will still affect the calculation of $\dot{T}$.

With $\dot{T}$ negative, the above equations indicate that both $\dot{\theta}_{max}$ and $\dot{\theta}_{min}$ are positive. Thus for $A > B > C$ (i.e., minimum moment of inertia about the spin axis) the spin axis deviation from an inertial reference increases with time and the body attitude is said to be unstable.

For the case $C > B > A$, $h^2 > 2TB$, which corresponds to a body spinning about its axis of largest inertia with a superimposed wobble, the equations for the rate of change of θ_{max} and θ_{min} can be shown to be identical to those for $A > B > C$ [Eqs. (9) and (10)]. With $C > B > A$ and $\dot{T}$ negative, the attitude angular rates $\dot{\theta}_{max}$ and $\dot{\theta}_{min}$ become negative, indicating a stable attitude condition where the spin axis approaches an inertially fixed direction.

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Flow Field of an Exhaust Plume Impinging on a Simulated Lunar Surface

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One of the problems associated with space exploration is determining the effects caused by the impingement of an exhaust plume on a foreign surface. Roberts, in a recent paper,[1] investigated the problem of a single axisymmetric jet impinging on a dust-covered lunar surface. He presented an approximate method for calculating the exhaust-plume flow field. However, this method is accurate only when the square of the nozzle exit Mach number is much greater than one ($M_j^2 \gg 1$) and the descent height is large. This note presents a less restricted method for calculating the exhaust-plume flow field and compares results with experimental data.

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Figure 1 shows a sketch of a lunar vehicle with its retrorocket impinging on a lunar surface. The exhaust gases of interest expand after leaving the nozzle and pass through a shock wave just above the lunar surface. Analytically the problem can be divided into the supersonic region ahead of the shock and the subsonic and supersonic regions behind it. The flow properties ahead of the shock can be calculated using the method of characteristics. The region behind it could probably best be calculated using an inverse method, similar to the one used to calculate the flow region around the nose of a blunt body. However, this would be difficult due to the high nonuniformity of the upstream flow. For now a simpler approach is preferable.

Newtonian theory as usually applied assumes a uniform upstream flow. However, for nonuniform flow it is still valid if applied locally. For this problem the upstream properties vary considerably with location, and, therefore, the local flow properties ahead of the shock must be determined. It is preferable to know the shock location although approximate results can be obtained by assuming that it lies on the lunar surface. Reference 1 presents a method for calculating the shock shape once the upstream flow field is known.

The flow properties on the lunar surface can then be calculated as follows:

1) The exhaust-plume flow field is calculated using the method of characteristics and assuming that the lunar surface is not present.

2) The shock location is calculated using the method presented in Ref. 1, or it is assumed to lie on the lunar surface.

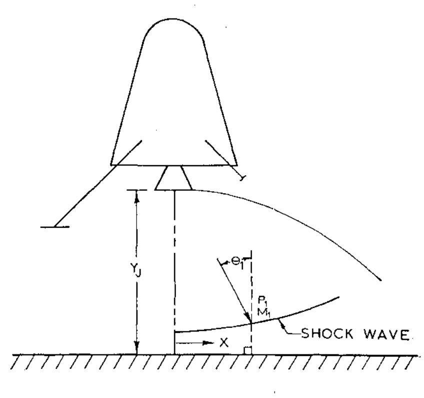
3) The flow properties just ahead of the shock are obtained using results from the method of characteristics.

4) The lunar-surface pressures are calculated using Newtonian theory,

$$p_s/P_c = (P_{T2}/p_1)(p_1/P_c)\cos^2\theta_1$$

where

p_1 = pressure just ahead of shock
p_s = lunar-surface static pressure
P_c = rocket-chamber total pressure
θ_1 = angle between the flow direction and the normal to the lunar surface
$\dfrac{P_{T2}}{p_1}$ = ratio of the total pressure behind a normal shock, at the point of interest, to the static pressure ahead of it



d_T = THROAT DIAMETER OF NOZZLE
Y_J = DISTANCE FROM NOZZLE EXIT
X = DISTANCE FROM NOZZLE CENTERLINE

Fig. 1 Lunar model